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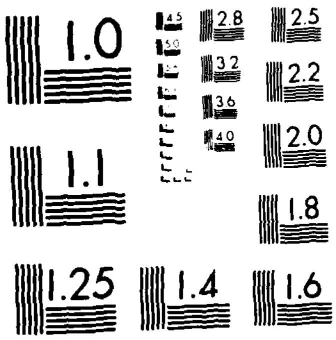
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# LIMIT THEOREMS FOR THE METHOD OF REPLICATION

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UNIVERSITY OF WISCONSIN - MADISON
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## ABSTRACT

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## SIGNIFICANCE AND EXPLANATION

Consider a stochastic system which is stable in the sense that it obeys an ergodic theorem. Simulators are frequently interested in estimating the steady-state ergodic limit. For example, in a queueing system, one often wants to analyze the long-run number of customers in queue. One popular method for estimating steady-state parameters involves generating independent replicates of the stochastic system; each replicate consists of the time evolution of the system up to some large (but finite) time horizon. In this paper, we obtain conditions under which this easily implemented method yields asymptotically valid confidence intervals for steady-state means.

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The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

LIMIT THEOREMS FOR THE METHOD OF REPLICATION

Peter W. Glynn

## 1. Introduction

Let $Y = \{Y(t) : t \geq 0\}$ be a real-valued stochastic process for which there exists a constant $r$ such that

$$\bar{Y}(t) \Rightarrow r$$

as $t \to \infty$, where $\bar{Y}(t) = t^{-1} \int_0^t Y(s)ds$ and $\Rightarrow$ denotes weak convergence. The parameter $r$ is called the steady-state mean of $Y$.

Simulators are frequently interested in estimating steady-state means. One popular procedure for accomplishing this, called the method of replication, involves simulating independent copies of the process $Y$. The independence of the copies (also known as replicates) simplifies construction of confidence intervals, since certain classical statistical procedures then become applicable; see LAW and KELTON (1982), Section 8.6, for a discussion of this approach to steady-state simulation. Our purpose, in this paper, is to determine precise conditions under which the method of replication is asymptotically valid.

One important feature of this study is that we allow schemes in which the number of replications is allowed to converge to infinity with the run-length of the simulation. This is done for two reasons. Firstly, it has been shown by Schmeiser [9] that in an asymptotic sense, confidence intervals with shortest expected length are obtained when one bases the intervals on limiting normal distributions (as opposed to Student - t statistics). As our paper indicates (see Proposition 3.1 and Corollary 3.9), the limiting normal appears only when the number of replicates is allowed to diverge to infinity.

Secondly, infinite replicate schemes allow the simulator to consistently estimate a parameter $\sigma^2$ (see Theorem 3.3), which is itself of some independent interest. The parameter $\sigma^2$ measures the asymptotic variability of $\bar{Y}(t)$ (see (2.2)); if $Y(t)$

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measures the cost of running a stochastic system at time $t$, then $r$ is the long-run average cost, and $\sigma^2$ measures the extent to which the average cost $\overline{Y}(t)$ may deviate from $r$ over the interval $[0,t]$. Thus, $\sigma^2$ may itself be important in determining the suitability of a policy to be evaluated over a planning horizon of $t$ time units duration.

## 2. A Central Limit Theorem for the Method of Replication

Let $\{Y_i : i \geq 1\}$ be a sequence of i.i.d. replicates of the continuous-time process $Y$. (In order to incorporate discrete-time stochastic sequences $U = \{U_n : n \geq 0\}$ into our framework, we set $Y(t) = U_{[t]}$, where $[t]$ denotes the greatest integer less than or equal to $t$.) A replication procedure is a non-decreasing function $m : \mathbb{R}^+ \to \mathbb{R}^+$ $(\mathbb{R}^+ = [0,\infty))$ for which there exists $k : \mathbb{R}^+ \to \{1,2,\ldots\}$ such that:

(2.1) i.) $m(t)\, k(t) \leq t$

ii.) $m(t)\, k(t)/t \to 1$ as $t \to \infty$

(if $m(t) > 0$, $k(t)$ can be taken to be $[t/m(t)]$). We interpret $k(t)$ as the number of replicates $Y_1,\ldots,Y_{k(t)}$, each simulated to time $m(t)$, associated with a computer budget of $t$ time units. Relation (2.1)i.) says that for each $t > 0$, the replication procedure must not use more time than is budgeted, whereas (2.1)ii.) requires that asymptotically one uses all the time allotted. The estimator $r(t)$ available after $t$ time units of computer time is given by

$$r(t) = \frac{1}{k(t)} \sum_{i=1}^{k(t)} \overline{Y}_i(m(t))$$

where $\overline{Y}_i(t) = t^{-1} \int_0^t Y_i(s)ds$.

As for the process $Y$, we shall assume throughout this paper that:

(2.2) i.) there exist constants $r$ and $0 < \sigma < \infty$ such that

$$X(t) \equiv t^{1/2}\,(\overline{Y}(t)-r) \Rightarrow \sigma N(0,1)$$

as $t \to \infty$, where $N(0,1)$ is a standard normal random variable (r.v.)

ii.) the process $\{X^2(t) : t \geq 0\}$ is uniformly integrable

iii.) $\sup\{t^{1/2}\,|EX(t)| : t \geq 0\} < \infty$.

Note that (2.2)i.) implies that $\overline{Y}(t) \Longrightarrow r$ as $t \to \infty$, so that the parameter $r$ is necessarily the steady-state mean of $Y$. Also, (2.2)ii.) is equivalent to requiring that

(2.3) $$v(t) \equiv EX^2(t) \to \sigma^2$$

as $t \to \infty$; see Theorem 5.4 of BILLINGSLEY [1]. Conditions which guarantee (2.2i.) and ii.) are available when $Y$ is a real-valued functional of a discrete-time Markov chain (see Theorems 1 and 3 in Section 16 of CHUNG [4]) or when $Y$ is a real-valued functional of a regenerative process (see Lemma 5 and Corollary 9.1 of SMITH [10]). A sufficient condition for (2.2)iii.) is provided by Theorem A, given in the Appendix.

Our first limit theorem shows that under a mild restriction on the replication procedure, $r(t)$ is an asymptotically normal estimator for $r$.

(2.4) THEOREM. Suppose that $m^2(t)/t \to \infty$ as $t \to \infty$. Then,

$$t^{1/2}(r(t) - r) \Longrightarrow \sigma N(0,1)$$

as $t \to \infty$.

PROOF. By Theorem 2.3 of BILLINGSLEY [1], it is sufficient to show that each sequence $t_n \to \infty$ *contains a subsequence* $t_n'$ such that

(2.5) $$(t_n')^{1/2}(r(t_n') - r) \Longrightarrow \sigma N(0,1)$$

as $n \to \infty$. There are three cases to consider:

case 1: $\sup\{k(t_n) : n \geq 1\} < \infty$: in this case, we can find a subsequence $t_n'$ such that $k(t_n') = k \geq 1$ for all $n$. Then,

(2.6) $$r(t_n') - r = \frac{1}{k}\sum_{i=1}^{k} (\overline{Y}_i(m(t_n')) - r) \quad .$$

By (2.2)i.) and the independence of the $Y_i$'s it follows that

(2.7) $$m(t_n')^{1/2}(\overline{Y}_1(m(t_n')) - r, \ldots, \overline{Y}_k(m(t_n')) - r) \Longrightarrow \sigma N(\vec{0}, I)$$

as $n \to \infty$, where $N(\vec{0},I)$ is a multivariate normal r.v. with mean vector $\vec{0}$ and covariance matrix $I$. By the continuous mapping principle (Theorem 5.1 of BILLINGSLEY [1]) applied to the continuous function $h(\vec{x}) = (x_1 + \ldots + x_k)/k$ ($\vec{x} = (x_1, \ldots, x_k)$), (2.6) and

(2.7) show that

$$(2.8) \qquad m(t_n')^{1/2}(r(t) - r) \Longrightarrow \sigma k^{-1/2} N(0,1)$$

as $n \to \infty$. But by (2.1)ii.), $k\, m(t_n')/t_n' \to 1$, so that (2.8) yields (2.5).

case 2: $\sup\{k(t_n) : n \geq 1\} = \infty$ and $\sigma > 0$: choose a subsequence $t_n'$ such that

$k(t_n') \to \infty$ as $n \to \infty$. Note that

$$(2.9) \qquad (t_n')^{1/2}(r(t_n') - r) = \alpha_n \sum_{i=1}^{k(t_n')} Z_{in} + \beta_n$$

where

$$Z_{in} = \gamma_n(\bar{Y}_i(m(t_n')) - Er(t_n'))$$

$$\gamma_n = m(t_n')^{1/2}/(k(t_n')v(m(t_n')))^{1/2}$$

$$\alpha_n = (t_n' v(m(t_n')))^{1/2}/(k(t_n')m(t_n'))^{1/2}$$

$$\beta_n = (t_n')^{1/2}(Er(t_n') - r) \quad .$$

Since $Er(t_n') = E\bar{Y}(m(t_n'))$, it follows that

$$\beta_n = (t_n'/m^2(t_n'))^{1/2} m(t_n')^{1/2} EX(m(t_n'))$$

and hence (2.2)iii.) shows that the assumption $m^2(t)/t \to \infty$ forces $\beta_n$ to tend to zero. Also, by (2.3) and (2.1)ii.), $\alpha_n \to \sigma$ as $n \to \infty$. To treat the sum in (2.9), we view the family of r.v.'s $\{Z_{in} : 1 \leq i \leq k(t_n'), n \geq 1\}$ as a triangular array. Note that $EZ_{in} = 0$ and

$$\sum_{i=1}^{k(t_n')} EZ_{in}^2 = 1 \quad .$$

Furthermore, by Chebyshev's inequality,

$$\max_{1 \leq i \leq k(t_n')} P\{Z_{in}^2 > \varepsilon\} \leq \frac{EZ_{in}^2}{\varepsilon} = \frac{1}{k(t_n')\varepsilon} \to 0$$

as $n \to \infty$, so we conclude that $\{Z_{in}\}$ is holospoudic (see p. 196-206 of CHUNG [5] for results and definitions). To show that

$$(2.10) \qquad \sum_{i=1}^{k(t_n')} Z_{in} \Longrightarrow N(0,1)$$

as $n \to \infty$, we need to verify Lindeberg's condition. Observe that for $\eta > 0$,

$$\sum_{i=1}^{k(t_n')} E\{Z_{in}^2 ; Z_{in}^2 > \eta\} \tag{2.11}$$

$$= k(t_n')E\{Z_{in}^2 ; Z_{in}^2 > \eta\} = E\{V_n^2 ; V_n^2 > k(t_n')\eta\}$$

where $V_n^2 = m(t_n')(\bar{Y}(m(t_n')) - Er(t_n'))^2/v(m(t_n'))$. But

$$V_n^2 < 2v(m(t_n'))^{-1}\{X^2(m(t_n')) + (EX(m(t_n')))^2\}$$

so that (2.2)ii.) and iii.) imply that $\{V_n^2\}$ is uniformly integrable; thus, (2.11) goes to zero as $n \to \infty$, verifying (2.10).

case 3: $\sup\{k(t_n) : n \geq 1\} = \infty$ and $\sigma = 0$ .

We shall reduce this case to the one above, in which $\sigma^2 > 0$. Let $\tilde{Y}$ be the process defined by

$$\tilde{Y}(t) = Y(t) + 2^{-1}N(0,1)t^{-1/2}$$

where $N(0,1)$ is independent of $Y$. Then,

$$\tilde{X}(t) = t^{1/2}(t^{-1}\int_0^t (\tilde{Y}(s) - r)ds) = X(t) + N(0,1)$$

satisfies (2.2)i.) - iii.). From case 2, it follows that

$$t^{1/2}(\tilde{r}(t) - r) \Longrightarrow N(0,1) \tag{2.12}$$

as $t \to \infty$. But the left-hand side of (2.12) has the distribution of

$$t^{1/2}(r(t) - r) + N(0,1) \tag{2.13}$$

where $N(0,1)$ is independent of $r(t)$. Letting $c(t,u)$ be the characteristic function of $t^{1/2}(r(t) - r)$, (2.12) and (2.13) prove that

$$c(t,u) \cdot e^{-u^2/2} \to e^{-u^2/2}$$

as $t \to \infty$, and hence $c(t,u) \to 1$ for all $u$. So,

$$t^{1/2}(r(t) - r) \Longrightarrow 0 \quad .$$

This theorem shows that the length of each replication should asymptotically dominate the total number of replicates, in the sense that $m(t)/k(t) \to \infty$ as $t \to \infty$.

## 3. Confidence Intervals for the Method of Replication.

Theorem 2.4 does not readily yield confidence intervals for $r$, since it involves the unknown constant $\sigma$. The type of confidence interval to be used depends on the behavior of $k(t)$.

(3.1) PROPOSITION. Suppose $\sigma^2 > 0$ and $k(t) \equiv k \geq 2$. Let

$$\Gamma(t) = \frac{1}{k(t)-1} \sum_{i=1}^{k(t)} (\overline{Y}_i(m(t)) - r(t))^2 .$$

Then,

$$k^{1/2}(r(t) - r)/\Gamma^{1/2}(t) \Longrightarrow t_{k-1} \tag{3.2}$$

as $t \to \infty$, where $t_{k-1}$ is a Student-t r.v. with $k - 1$ degrees of freedom.

PROOF. For any sequence $t_n'$ converging to infinity, relation (2.7) holds. Applying the continuous mapping principle to the function

$$h(\vec{x}) = k^{1/2}\overline{x}_k\Big(\frac{1}{k-1} \sum_{i=1}^{k} (x_i - \overline{x}_k)^2\Big)^{-1/2} ,$$

where $\overline{x}_k = (x_1 + \ldots + x_k)/k$ (note that $h$ is appropriately continuous since $\sigma^2 > 0$), we find that

$$k^{1/2}(r(t_n') - r)/\Gamma^{1/2}(t_n') \Longrightarrow t_{k-1} ;$$

this proves (3.2).

(3.3) THEOREM. Assume that the process $\{X^4(t) : t \geq 0\}$ is uniformly integrable. If $k(t) \to \infty$ as $t \to \infty$ and $m^2(t)/t \to \infty$ as $t \to \infty$, then

$$m(t)\Gamma(t) \Longrightarrow \sigma^2$$

as $t \to \infty$.

PROOF. Note that

$$\begin{aligned} \frac{k(t)-1}{k(t)} m(t)\Gamma(t) &= \frac{1}{k(t)} \sum_{i=1}^{k(t)} m(t)(\overline{Y}_i(m(t)) - r(t))^2 \\ &= \frac{1}{k(t)} \sum_{i=1}^{k(t)} m(t)(\overline{Y}_i(m(t)) - r)^2 \\ &\quad - m(t)(r(t) - r)^2 . \end{aligned} \tag{3.4}$$

By Theorem 2.4, $t(r(t) - r)^2 \Longrightarrow \sigma^2 N(0,1)^2$ as $t \to \infty$ (continuous mapping with $h(x) = x^2$), so

$$m(t)(r(t) - r)^2 = \frac{m(t)}{t} \cdot t(r(t) - r)^2 \Longrightarrow 0 \tag{3.5}$$

as $t \to \infty$, by (2.1)ii.) and the fact that $k(t) \to \infty$. As for the other term in (3.4), note that

$$m(t)E(\overline{Y}_i(m(t)) - r)^2 = v(t) \tag{3.6}$$

and

$$\begin{aligned} &\operatorname{var}\Big(\frac{1}{k(t)} \sum_{i=1}^{k(t)} m(t)(\overline{Y}_i(m(t)) - r)^2\Big) \\ &= \frac{1}{k(t)} \{E|X(t)|^4 - v^2(t)\} \quad . \end{aligned} \tag{3.7}$$

Since $\{X^4(t) : t > 0\}$ is uniformly integrable, it follows that $EX^4(\cdot)$ is a bounded function (see (5.1) of [1]) so that (2.3) implies that the variance term (3.7) tends to zero as $t \to \infty$. By Chebyshev's inequality,

$$\begin{aligned} &P\Big\{\Big|\frac{1}{k(t)} \sum_{i=1}^{k(t)} m(t)(\overline{Y}_i(m(t)) - r)^2 - v(t)\Big| > \varepsilon\Big\} \\ &\leq \operatorname{var}\Big(\frac{1}{k(t)} \sum_{i=1}^{k(t)} m(t)(\overline{Y}_i(m(t)) - r)^2\Big) \to 0 \end{aligned}$$

as $t \to \infty$, by (3.6) and (3.7). Thus,

$$\frac{1}{k(t)} \sum_{i=1}^{k(t)} m(t)(\overline{Y}_i(m(t)) - r)^2 - v(t) \Longrightarrow 0 \tag{3.8}$$

as $t \to \infty$. Combining (2.3), (3.4), (3.5), and (3.8) yields the result.

The following corollary is immediate from Theorems 2.4 and 3.3.

(3.9) COROLLARY. Under the assumptions of Theorem 3.3,

$$t^{1/2}(r(t) - r)/(m(t)\Gamma(t))^{1/2} \Longrightarrow N(0,1) \tag{3.10}$$

as $t \to \infty$ provided $\sigma^2 > 0$.

Generation of confidence intervals for $r$, based on the limit theorems (3.2) and (3.10), is straightforward. A condition guaranteeing that $\{X^4(t) : t > 0\}$ is uniformly integrable is given by Theorem B in the Appendix.

Appendix

Let $Y$ be a non-delayed regenerative process with regeneration times $0 = T(0) < T(1) < \dots$ . (See p. 298-302 of ÇINLAR [6]) for a discussion of regenerative processes.) Assume that

$$E\left(\int_0^{T(1)} (1 + |Y(s)|)ds\right)^8 < \infty \quad . \tag{A1}$$

Then, SMITH [10] proved that $Y$ satisfies (1.2)i.) and ii.) and that the steady-state mean has the representation

$$r = E\left(\int_0^{T(1)} Y(s)ds\right)/ET(1) \quad . \tag{A2}$$

Set $\tau_k = T(k) - T(k-1)$, $N(t) = \max\{k \geq 0 : T(k) \leq t\}$, and

$$Z_i = \int_{T(i-1)}^{T(i)} (Y(s) - r)ds \quad .$$

THEOREM A. Under (A1), (2.2)iii.) is satisfied.

PROOF. It is easy to see that

$$\int_0^t Y(s)ds - rt = \sum_{k=1}^{N(t)+1} Z_k - \int_t^{T(N(t)+1)} (Y(s) - r)ds \quad . \tag{A3}$$

From (A1), it is obvious that $E(|Z_1| + \tau_1) < \infty$ and thus Wald's identity implies that

$$E\left(\sum_{k=1}^{N(t)+1} Z_k\right) = E(N(t)+1) \cdot EZ_k \quad .$$

(see p. 137 of CHUNG [5].) By the definition of $Z_k$, (A2) proves that $EZ_k = 0$ and thus the first term on the right-hand side of (A3) vanishes. For the second term, set

$$a(t) = E\left(\int_t^{T(N(t)+1)} (Y(s) - r)ds\right) \quad . \tag{A4}$$

A simple renewal argument shows that $a$ satisfies the renewal equation

$$a(t) = b(t) + (a * F)(t)$$

where $*$ denotes convolution and

$$b(t) = E\{\int_t^{T(1)} (Y(s) - r)ds \; ; \; T(1) > t\}$$

$$F(t) = P\{T(1) \leq t\} \quad .$$

Since

$$a(t) \leq E\left(\sum_{k=1}^{N(t)+1} \int_{T(k-1)}^{T(k)} |Y(s) - r|ds\right)$$

$$\leq E(N(t) + 1) \cdot E\left(\int_0^{T(1)} |Y(s) - r|ds\right) < \infty$$

by Wald's equality and (A1), it is clear that $a(\cdot)$ is bounded over finite intervals. Hence, by Theorem 2.3 of ÇINLAR [6],

(A5) $$a(t) = (b * U)(t)$$

where $U(t) = \sum_{k=0}^{\infty} P\{T(k) \leq t\} = EN(t) + 1$. But

$$b(t) \leq E\{\int_t^{T(1)} (|Y(s)| + r)ds \quad ; \quad T(1) > t\}$$

$$\leq \frac{1}{t^2} E\{T(1)^2 \int_t^{T(1)} (|Y(s)| + r)ds \quad ; \quad T(1) > t\}$$

$$\leq \frac{1}{t^2} E\{T(1)^2 \int_0^{T(1)} |Y(s)|ds + rT(1)^3\}$$

and thus $b(t) \leq c_1 \min\{1, t^{-2}\}$ for some $c_1$. So, there exists $\alpha$ and $c_2$ such that

(A6)
$$|a(t)| = \left|\int_{[0,t]} b(t-s)U(ds)\right|$$

$$\leq \int_{(t-\alpha,t]} c_2 \, U(ds) + \int_{[0,t-\alpha]} c_2 \, t^{-1}U(ds$$

$$\leq c_2(U(t) - U(t-\alpha)) + c_2 \, U(t)/t \quad ;$$

the first term above is bounded in $t$ by relation (1.7), p. 360, of FELLER [7], whereas the second is bounded by the elementary renewal theorem (see Theorem 5.5.2 of CHUNG [5]). From (A4) and (A5), it is evident from (A3) that

$$E\left(\int_0^t Y(s)ds - rt\right)$$

is a bounded function of $t$, proving the theorem.

THEOREM B. Assume (A1) holds. Then,

$$EX(t)^4 \to 3\sigma^4$$

as $t \to \infty$, and $\{X^4(t) : t \geq 0\}$ is uniformly integrable.

PROOF. Note that

$$M(n) = \sum_{k=1}^{n} Z_k$$

is a martingale. For $t$ fixed, set $M_n = M((N(t)+1) \wedge n)$, where $a \wedge b \equiv \min(a,b)$. Since $N(t) + 1$ is a stopping time, it follows that $\{M_n : n \geq 1\}$ is also a martingale. (See Proposition 5.26 of BREIMAN [2].)

Let $D_n = M_n - M_{n-1} = Z_n I\{N(t)+1 \geq n\}$. By Burkholder's [3] square function inequality for martingales

$$\text{(A7)} \qquad \max_{1 \leq k < \infty} EM_k^6 \leq c_6 E\Big(\sum_{j=1}^{\infty} D_j^2\Big)^3 = c_6 E\Big(\sum_{j=1}^{N(t)+1} Z_j^2\Big)^3$$

where $c_k = 18(k^{3/2}/(k-1))^k$. Note that since $N(t) < \infty$ a.s., $M_n \to M(N(t)+1)$ a.s. as $n \to \infty$. By (A7) and Fatou's lemma,

$$\text{(A8)} \qquad EM(N(t)+1)^6 \leq \varliminf_{n \to \infty} EM_n^6 \leq c_6 E\Big(\sum_{j=1}^{N(t)+1} Z_j^2\Big)^3 .$$

Setting $U_j = Z_j^2 - EZ_1^2$, observe that Minkowski's inequality yields

$$\text{(A9)} \qquad E\Big(\sum_{j=1}^{N(t)+1} Z_j^2\Big)^3 = E\Big(\sum_{j=1}^{N(t)+1} U_j + EZ_1^2 \cdot (N(t)+1)\Big)^3$$

$$\leq \Big(E^{1/3}\Big|\sum_{j=1}^{N(t)+1} U_j\Big|^3 + E^{1/3} Z_1^2 \cdot E^{1/3}(N(t)+1)^3\Big)^3$$

$$\leq 8 \max\Big(E\Big|\sum_{j=1}^{N(t)+1} U_j\Big|^3, EZ_1^2 \cdot E(N(t)+1)^3\Big) .$$

Applying the Burkholder inequality a second time, we find that

$$\text{(A10)} \qquad E\Big|\sum_{j=1}^{N(t)+1} U_j\Big|^3 \leq c_3 E\Big(\sum_{j=1}^{N(t)+1} U_j^2\Big)^{3/2} \leq c_3\Big(1 + E\Big(\sum_{j=1}^{N(t)+1} U_j^2\Big)^2\Big) .$$

Lemma 5 of SMITH [10] proves that under (A1),

$$\operatorname{var}\Big(\sum_{j=1}^{N(t)+1} U_j^2\Big)^2 = O(t) \quad ;$$

Wald's equality therefore implies that the right-hand side of (A10) is $O(t^2)$.

By Lemma A12, we conclude that the bound in (A9) is $O(t^3)$, so that (A8) proves that

(A11) $$EN(N(t)+1)^6 = E\Big(\sum_{j=1}^{N(t)+1} Z_j\Big)^6 = O(t^3) \quad .$$

Furthermore, we have that

$$E\Big(\int_t^{T(N(t)+1)} (Y(s) - r)ds\Big)^6 \le E\Big(\sum_{j=1}^{N(t)+1} \Big(\int_{T(j-1)}^{T(j)} |Y(s) - r|ds\Big)^6\Big)$$

which is $O(t)$ by Wald's equality and (A1). So, by Minkowski's inequality and (A3), we find that

$$\sup_t EX(t)^6 < \infty \quad ;$$

thus $\{X^4(t) : t \ge 0\}$ is uniformly integrable and $EX^4(t) \to \sigma^4 EN(0,1)^4 = 3\sigma^4$ as $t \to \infty$.

(A12). LEMMA. If $ET(1)^m < \infty$ for some $m \ge 1$, then $E(N(t) + 1)^m = O(t^m)$ as $t \to \infty$.

PROOF. It is well known that for any non-negative r.v. $N$, $EN^m$ is bounded by a multiple of

(A13) $$\sum_{k=1}^{\infty} k^{m-1} P\{N \ge k\} \quad .$$

Since $\{N(t) + 1 \ge k\} = \{T(k) \le t\}$, (A13) is bounded by

(A14) $$\sum_{k \le 2t/\mu} k^{m-1} + \sum_{k > 2t/\mu} k^{m-1} P\{\hat{T}(k) \le t - k\mu\}$$

$(\mu = ET(1),\ \hat{T}(k) = T(k) - k\mu)$; comparison of the first term in (A14) with the integral of $g(x) = x^{m-1}$ shows that it is of order $O(t^m)$. For the second term, observe that if $k > 2t/\mu$, then $\mu k/2 < \mu k - t$ so that term is dominated by

(A15) $$\sum_{k > 2t/\mu} k^{m-1} \cdot P\{\hat{T}(k) \ge \mu k/2\} \quad .$$

By Chebyshev's inequality and (A1), the probability in (A15) is bounded by $E\hat{T}(k)^8/(\mu k/2)^8$, which is $O(k^{-4})$. (since $E\hat{T}(k)^8 = O(k^4)$, as may be verified algebraically.) Thus, (A15) is summable and bounded in $t$; (A14) then yields our result.

It is worth noting that (A1) is automatically satisfied when $Y$ corresponds to an irreducible finite state Markov chain, in either discrete or continuous time.

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The method of replication is frequently used by simulators to estimate steady-state quantities. In this paper, we obtain conditions under which this method yields asymptotically valid confidence intervals for steady-state means.

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